Learning Objectives

- Use Riemann Sums to approximate areas under curves
- Evaluate definite integrals as limits of Riemann Sums

Introduction

In the Lesson The Area Problem we defined the area under a curve in terms of a limit of sums.

$$A=\lim_{n\to+\infty}S(P)=\lim_{n\to+\infty}T(P)$$

where

$$S(P)=\sum_1^n m_i(x_i-x_{i-1})=m_1(x_1-x_0)+m_2(x_2-x_1)+\ldots+m_n(x_n-x_{n-1}),$$
$$T(P)=\sum_1^n M_i(x_i-x_{i-1})=M_1(x_1-x_0)+M_2(x_2-x_1)+\ldots+M_n(x_n-x_{n-1}),$$

$S(P)$, and $T(P)$ were examples of ***Riemann Sums***. In general, Riemann Sums are of form $\sum_{i=1}^n f(x_i^*)\Delta x$ where each x_i^* is the value we use to find the length of the rectangle in the i^{th} sub-interval. For example, we used the maximum function value in each sub-interval to find the upper sums and the minimum function in each sub-interval to find the lower sums. But since the function is continuous, we could have used any points within the sub-intervals to find the limit. Hence we can define the most general situation as follows:

Definition

> If f is continuous on $[a,b]$, we divide the interval $[a,b]$ into n sub-intervals of equal width with $\Delta x=\frac{b-a}{n}$. We let $x_0=a,x_1,x_2,\ldots,x_n=b$ be the endpoints of these sub-intervals and let $x_1^*,x_2^*,\ldots,x_n^*$ be *any* sample points in these sub-intervals. Then the ***definite integral*** of f from $x=a$ to $x=b$ is

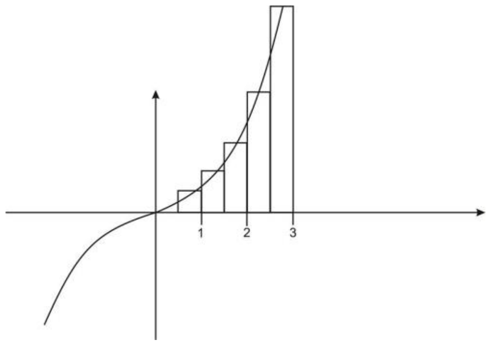
$$\int_a^b f(x)dx=\lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x.$$

Example 1:

Evaluate the Riemann Sum for $f(x)=x^3$ from $x=0$ to $x=3$ using $n=6$ sub-intervals and taking the sample points to be the midpoints of the sub-intervals.

Solution:

If we partition the interval $[0,3]$ into $n=6$ equal sub-intervals, then each sub-interval will have length $\frac{3-0}{6}=\frac{1}{2}$. So we have $\Delta x=\frac{1}{2}$ and



$R_6=\sum_1^6 f(x_i^*)\Delta x=f(0.25)\Delta x+f(0.75)\Delta x+f(1.25)\Delta x+f(1.75)\Delta x+f(2.25)\Delta x+f(2.75)\Delta x$
$=(164)(12)+(2764)(12)+(12564)(12)+(34364)(12)+(72964)(12)+(133164)(12)$
$=255664=39.93.$

Now let's compute the definite integral using our definition and also some of our summation formulas.

Example 2:

Use the definition of the definite integral to evaluate $\int_0^3 x^3 dx$.
Solution:

Applying our definition, we need to find

$\int_0^3 x^3 dx=\lim_{n\to\infty}\sum_{i=1}^n f(x_i^*)\Delta x.$

We will use right endpoints to compute the integral. We first need to divide $[0,3]$ into n sub-intervals of length $\Delta x=3-0n=3n$. Since we are using right endpoints, $x_0=0, x_1=3n, x_2=6n, \ldots x_i=3in$.
So $\int_0^3 x^3 dx= \lim_{n\to\infty}\sum_{i=1}^n f(3in)(3n)=\lim_{n\to\infty}3n\sum_{i=1}^n(3in)^3=\lim_{n\to\infty}3n\sum_{i=1}^n(27n^3)i^3=\lim_{n\to\infty}81n^4\sum_{i=1}^n i^3$.
Recall that $\sum_1^n i^3=[n(n+1)2]^2$. By substitution, we have
$\int_0^3 x^3 dx=\lim_{n\to\infty}=81n^4[n(n+1)2]^2=\lim_{n\to\infty}814[1+1n]^2\to 814 \text{ as } n\to\infty.$
Hence

$\int_0^3 x^3 dx=814.$

Before we look to try some problems, let's make a couple of observations. First, we will soon not need to rely on the summation formula and Riemann Sums for actual computation of definite integrals. We will develop several computational strategies in order to solve a variety of problems that come up. Second, the idea of definite integrals as approximating the area under a curve can be a bit confusing since we may sometimes get results that do not make sense when interpreted as areas. For example, if we were to compute the definite integral $\int_{-3}^3 x^3 dx$,then due to the symmetry of $f(x)=x^3$ about the origin, we would find that $\int_{-3}^3 x^3 dx=0$. This is because for every sample point x_j^*, we also have $-x_j^*$ is also a sample point with $f(-x_j^*)=-f(x_j^*)$. Hence, it is more accurate to say that $\int_{-3}^{3} x^3 dx$ gives us the ***net area*** between $x=-3$ and $x=3$. If we wanted the ***total area*** bounded by the graph and the $x-$axis, then we would compute $2\int_{0}^{3} x^3 dx = 81/2$.

Lesson Summary

1. We used Riemann Sums to approximate areas under curves.
2. We evaluated definite integrals as limits of Riemann Sums.

The following applet lets you explore Riemann Sums of any function. You can change the bounds and the number of partitions. Follow the examples given on the page, and then use the applet to explore on your own. Riemann Sums Applet. Note: On this page the author uses Left- and Right- hand sums. These are similar to the sums $S(P)$ and $T(P)$ that you have learned, particularly in the case of an increasing (or decreasing) function. Left-hand and Right-hand sums are frequently used in calculations of numerical integrals because it is easy to find the left and right endpoints of each interval, and much more difficult to find the max/min of the function on each interval. The difference is not always important from a numerical approximation standpoint; as you increase the number of partitions, you should see the Left-hand and Right-hand sums converging to the same value. Try this in the applet to see for yourself.

Review Questions

In problems #1–7 , use Riemann Sums to approximate the areas under the curves.

1. Consider $f(x)=2-x$ from $x=0$ to $x=2$. Use Riemann Sums with four subintervals of equal lengths. Choose the midpoints of each subinterval as the sample points.
2. Repeat problem #1 using geometry to calculate the exact area of the region under the graph of $f(x)=2-x$ from $x=0$ to $x=2$. (Hint: Sketch a graph of the region and see if you can compute its area using area measurement formulas from geometry.)
3. Repeat problem #1 using the definition of the definite integral to calculate the exact area of the region under the graph of $f(x)=2-x$from $x=0$ to $x=2$.
4. $f(x)=x^2-x$ from $x=1$ to $x=4$. Use Riemann Sums with five subintervals of equal lengths. Choose the left endpoint of each subinterval as the sample points.
5. Repeat problem #4 using the definition of the definite intergal to calculate the exact area of the region under the graph of $f(x)=x^2-x$ from $x=1$ to $x=4$.
6. Consider $f(x)=3x^2$. Compute the Riemann Sum of f on $[0,1]$under each of the following situations. In each case, use the right endpoint as the sample points.
   a. Two sub-intervals of equal length.
   b. Five sub-intervals of equal length.
   c. Ten sub-intervals of equal length.

d. Based on your answers above, try to guess the exact area under the graph of f on $[0,1]$.

7. Consider $f(x)=e^x$. Compute the Riemann Sum of f on $[0,1]$ under each of the following situations. In each case, use the right endpoint as the sample points.
   a. Two sub-intervals of equal length.

   b. Five sub-intervals of equal length.

   c. Ten sub-intervals of equal length.

   d. Based on your answers above, try to guess the exact area under the graph of f on $[0,1]$.
8. Find the net area under the graph of $f(x)=x^3-x$; $x=-1$ to $x=1$. (Hint: Sketch the graph and check for symmetry.)
9. Find the total area bounded by the graph of $f(x)=x^3-x$ and the $x-$axis, from to $x=-1$ to $x=1$.
10. Use your knowledge of geometry to evaluate the following definite integral: $\int_0^3 \sqrt{9-x^2}\,dx$ (Hint: set $y=\sqrt{9-x^2}$ and square both sides to see if you can recognize the region from geometry.)