## Learning Objectives

- Use antiderivatives to evaluate definite integrals
- Use the Mean Value Theorem for integrals to solve problems
- Use general rules of integrals to solve problems

## Introduction

In the Lesson on Definite Integrals, we evaluated definite integrals using the limit definition. This process was long and tedious. In this lesson we will learn some practical ways to evaluate definite integrals. We begin with a theorem that provides an easier method for evaluating definite integrals. Newton discovered this method that uses antiderivatives to calculate definite integrals.

**Theorem 4.1:**

If $f$ is continuous on the closed interval $[a,b]$, then

$$\int_a^b f(x)dx=F(b)-F(a),$$

where $F$ is any antiderivative of $f$.

We sometimes use the following shorthand notation to indicate $\int_a^b f(x)dx=F(b)-F(a)$:

$$\int_a^b f(x)dx=F(x)\Big]_a^b.$$

The proof of this theorem is included at the end of this lesson. Theorem 4.1 is usually stated as a part of the Fundamental Theorem of Calculus, a theorem that we will present in the Lesson on the Fundamental Theorem of Calculus. For now, the result provides a useful and efficient way to compute definite integrals. We need only find an antiderivative of the given function in order to compute its integral over the closed interval. It also gives us a result with which we can now state and prove a version of the Mean Value Theorem for integrals. But first let’s look at a couple of examples.

**Example 1:**

Compute the following definite integral:

$$\int_0^3 x^3dx.$$

**Solution:**

Using the limit definition we found that $\int_0^3 x^3dx=\frac{81}{4}$. We now can verify this using the theorem as follows:

We first note that $x^4/4$ is an antiderivative of $f(x)=x^3$. Hence we have

$$\int_0^3 x^3dx=\frac{x^4}{4}\Big]_0^3=\frac{81}{4}-\frac{0}{4}=\frac{81}{4}.$$

We conclude the lesson by stating the rules for definite integrals, most of which parallel the rules we stated for the general indefinite integrals.

$\int_a^a f(x)dx \int_a^b f(x)dx \int_a^b k\cdot f(x)dx \int_a^b [f(x)\pm g(x)]dx \int_a^b f(x)dx = 0 = -\int_b^a f(x)dx = k\int_a^b f(x)dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where $a<c<b$.

Given these rules together with Theorem 4.1, we will be able to solve a great variety of definite integrals.

**Example 2:**

Compute $\int_{-2}^{2}(x-\sqrt{x})dx$.

**Solution:**

$\int_1^4 (x-\sqrt{x})dx = \int_1^4 x\,dx - \int_1^4 \sqrt{x}\,dx = \frac{x^2}{2}\Big]_1^4 - \frac{2}{3}x^{\frac{3}{2}}\Big]_1^4 = (8-\frac{1}{2}) - \frac{2}{3}(8-1) = \frac{15}{2} - \frac{14}{3} = \frac{17}{6}$.

**Example 3:**

Compute $\int_0^{\frac{\pi}{2}}(x+\cos x)dx$.

**Solution:**

$\int_0^{\frac{\pi}{2}}(x+\cos x)dx = \int_0^{\frac{\pi}{2}}(x)dx = \int_0^{\frac{\pi}{2}}(\cos x)dx = \frac{x^2}{2}\Big|_0^{\frac{\pi}{2}} + \frac{\sin x}{1}\Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8} + 1 = \frac{\pi^2+8}{8}$.

## Lesson Summary

1. We used antiderivatives to evaluate definite integrals.
2. We used the Mean Value Theorem for integrals to solve problems.
3. We used general rules of integrals to solve problems.

**Proof of Theorem 4.1**

We first need to divide $[a,b]$ into $n$ sub-intervals of length $\Delta x = \frac{b-a}{n}$. We let $x_0=a, x_1, x_2, \ldots, x_n=b$ be the endpoints of these sub-intervals.

Let $F$ be any antiderivative of $f$.

Consider $F(b)-F(a)=F(x_n)-F(x_0)$.

We will now employ a method that will express the right side of this equation as a Riemann Sum. In particular,

$F(b)-F(a)=F(x_n)-F(x_0)=F(x_n)-F(x_{n-1})+F(x_{n-1})-F(x_{n-2})+F(x_{n-2})-\ldots+F(x_1)-F(x_0)=\sum_1^n[F(x_i)-F(x_{i-1})]$.

Note that $F$ is continuous. Hence, by the Mean Value Theorem, there exist $c_i\in[x_i-x_{i-1},x_i]$ such that $F(x_i)-F(x_{i-1})=F'(c_i)(x_i-x_{i-1})=f(c_i)\Delta x$.

Hence

$F(b)-F(a)=\sum_1^n F'(c_i)(x_i-x_{i-1})=\sum_1^n f(c_i)\Delta x$.

Taking the limit of each side as $n\to\infty$ we have

$\lim_{n\to\infty}[F(b)-F(a)]=\lim_{n\to\infty}\sum_{1}^{n} f(c_i)\Delta x.$

We note that the left side is a constant and the right side is our definition for $\int_a^b f(x)dx.$ Hence

$$F(b)-F(a)=\lim_{n\to\infty}\sum_{1}^{n} f(c_i)\Delta x=\int_a^b f(x)dx.$$

**Proof of Theorem 4.2**

Let $F(x)=\int_a^x f(x)dx.$

By the Mean Value Theorem for derivatives, there exists $c\in[a,b]$ such that

$F'(c)=\frac{F(b)-F(a)}{b-a}.$

From Theorem 4.1 we have that $F$ is an antiderivative of $f$. Hence, $F'(x)=f(x)$ and in particular, $F'(c)=f(c)$. Hence, by substitution we have

$f(c)=\frac{F(b)-F(a)}{b-a}.$

Note that $F(a)=\int_a^a f(x)dx=0.$ Hence we have

$f(c)=\frac{F(b)-0}{b-a}=\frac{F(b)}{b-a},$

and by our definition of $F(x)$ we have

$f(c)=\frac{1}{b-a}F(b)=\frac{1}{b-a}\int_a^b f(x)dx.$

This theorem allows us to find for positive functions a rectangle that has base $[a,b]$ and height $f(c)$ such that the area of the rectangle is the same as the area given by $\int_a^b f(x)dx.$ In other words, $f(c)$ is the average function value over $[a,b]$.

## Review Questions

In problems #1–8, use antiderivatives to compute the definite integral.

1. $\int_4^9 (3\sqrt{x})dx$
2. $\int_0^1 (t-t^2)dt$
3. $\int_2^5 \left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{2}}\right)dx$
4. $\int_0^1 4(x^2-1)(x^2+1)dx$
5. $\int_2^8 (4x+x^2+x)dx$
6. $\int_2^4 (e^{3x})dx$
7. $\int_1^4 2x+3dx$
8. Find the average value of $f(x)=\sqrt{x}$ over $[1,9]$.
9. If $f$ is continuous and $\int_1^4 f(x)dx=9$, show that $f$ takes on the value $3$ at least once on the interval $[1,4]$.
10. Your friend states that there is no area under the curve of $f(x)=\sin x$on $[0,2\pi]$ since he computed $\int_0^{2\pi}\sin x dx=0.$ Is he correct? Explain your answer.