Learning Objectives

A student will be able to:

- Compute by hand the integrals of a wide variety of functions by using technique of Trigonometric Substitution.
- Combine this technique with other integration techniques to integrate.

When we are faced with integrals that involve radicals of the forms $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, and $\sqrt{x^2+a^2}$, we may make substitutions that involve trigonometric functions to eliminate the radical. For example, to eliminate the radical in the expression

$\sqrt{a^2-x^2}$

we can make the substitution

$x=a\sin\theta$,

$-\pi/2\le\theta\le\pi/2$,

(Note: θ must be limited to the range of the inverse sine function.)

which yields,

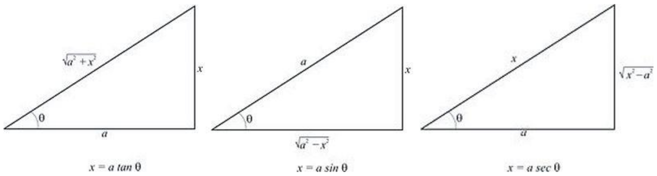
$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2\theta}=\sqrt{a^2(1-\sin^2\theta)}=\sqrt{a\cos^2\theta}=a\cos\theta.$$

The reason for the restriction $-\pi/2\le\theta\le\pi/2$ is to guarantee that $\sin\theta$ is a one-to-one function on this interval and thus has an inverse.

The table below lists the proper trigonometric substitutions that will enable us to integrate functions with radical expressions in the forms above.

Expression in Integrand	Substitution	Identity Needed
$\sqrt{a^2-x^2}$	$x=a\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
$\sqrt{a^2+x^2}$	$x=a\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
$\sqrt{x^2-a^2}$	$x=a\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

In the second column are listed the most common substitutions. They come from the reference right triangles, as shown in the figure below. We want any of the substitutions we use in the integration to be reversible so we can change back to the original variable afterward. The right triangles in the figure below will help us reverse our substitutions.



Description: 3 triangles.

Example 1:

Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$.
Solution:

Our goal first is to eliminate the radical. To do so, look up the table above and make the substitution

$$x=2\sin\theta, -\pi/2\le\theta\le\pi/2,$$
so that

$$\frac{dx}{d\theta}=2\cos\theta$$
Our integral becomes

$$\int \frac{dx}{x^2\sqrt{4-x^2}}=\int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2\sqrt{4-4\sin^2\theta}}=\int \frac{2\cos\theta\, d\theta}{(2\sin\theta)^2(2\cos\theta)}=\frac{1}{4}\int \frac{d\theta}{\sin^2\theta}=\frac{1}{4}\int \csc^2\theta\, d\theta=-\frac{1}{4}\cot\theta+C.$$
Up to this stage, we are done integrating. To complete the solution however, we need to express $\cot\theta$ in terms of x. Looking at the figure of triangles above, we can see that the second triangle represents our case, with $a=2$. So $x=2\sin\theta$ and $2\cos\theta=\sqrt{4-x^2}$, thus
$$\cot\theta=\frac{\sqrt{4-x^2}}{x},$$
since

$$\cot\theta=\frac{\cos\theta}{\sin\theta}.$$
so that

$$\int \frac{dx}{x^2\sqrt{4-x^2}}=-\frac{1}{4}\cot\theta+C=-\frac{1}{4}\frac{\sqrt{4-x^2}}{x}+C.$$
Example 2:

Evaluate $\int \frac{\sqrt{x^2-3}}{x}dx$.
Solution:

Again, we want to first to eliminate the radical. Consult the table above and substitute $x=\sqrt{3}\sec\theta$. Then $dx=\sqrt{3}\sec\theta\tan\theta\, d\theta$. Substituting back into the integral,
$$\int \frac{\sqrt{x^2-3}}{x}dx=\int \frac{\sqrt{3\sec^2\theta-3}}{\sqrt{3}\sec\theta}\sqrt{3}\sec\theta\tan\theta\, d\theta=\sqrt{3}\int \tan^2\theta\, d\theta.$$
Using the integral identity from the section on Trigonometric Integrals,

$\int \tan^m x dx = \tan^{m-1} x m-1 - \int \tan^{m-2} x dx$.
and letting $m=2$ we obtain
$\int x^2-3-----\surd x dx = 3-\surd((\tan\theta)-\theta)+C$.
Looking at the triangles above, the third triangle represents our case, with $a=3-\surd$.
So $x=3-\surd\sec\theta$ and thus $\cos x = 3-\surd/x$, which gives $\tan\theta = x^2-3-----\surd/3-\surd$.
Substituting,
$\int x^2-3-----\surd x dx = 3-\surd((\tan\theta)-\theta)+C = 3-\surd(x^2-3-----\surd 3-\surd - \tan^{-1}(x^2-3-----\surd 3-\surd))+C = x^2-3-----\surd - 3-\surd\tan^{-1}(x^2-3-----\surd 3-\surd)+C$.

Example 3:

Evaluate $\int dx x^2 x^2+1-----\surd$.

Solution:

From the table above, let $x=\tan\theta$ then $dx=\sec^2\theta d\theta$. Substituting into the integral,
$\int dx x^2 x^2+1-----\surd = \int \sec^2\theta d\theta \tan^2\theta \tan^2\theta+1--------\surd$.
But since $\tan^2\theta+1=\sec^2\theta$,
$=\int \sec^2\theta d\theta \tan^2\theta \sec\theta = \int \sec\theta \tan^2\theta d\theta = \int 1 \cos\theta \cos^2\theta \sin^2\theta d\theta = \int \cot\theta \csc\theta d\theta$.
Since $d d\theta(\csc\theta) = -\cot\theta\csc\theta$,
$\int dx x^2 x^2+1-----\surd = \int \cot\theta\csc\theta d\theta = -\csc\theta + C$.
Looking at the triangles above, the first triangle represents our case, with $a=1$. So $x=\tan\theta$ and thus $\sin x = x 1+x^2-----\surd$, which gives $\csc\theta = 1+x^2-----\surd x$. Substituting,
$\int dx x^2 x^2+1-----\surd = -\csc\theta + C = -1+x^2-----\surd x + C$.

Review Questions

Evaluate the integrals.

1. $\int 4-x^2-----\surd dx$
2. $\int 1 9+x^2-----\surd dx$
3. $\int x^3 1-x^2-----\surd dx$
4. $\int 1 1-9x^2------\surd dx$
5. $\int x^3 4-x^2-----\surd dx$
6. $\int 1 x^2 x^2-36------\surd dx$
7. $\int 1 (x^2+25)^2 dx$
8. $\int_4^0 x^3 16-x^2------\surd dx$
9. $\int_{-\pi}^0 e^x 1-e^{2x}------\surd dx$ (*Hint:* First use u−substitution, letting $u=e^x$)
10. Graph and then find the area of the surface generated by the curve $y=x^2$ from $x=1$ to $x=0$ and revolved about the x−axis.