4.7 Integration by Substitution

1. Let $u = \ln x$ and $dv = x$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^2}{2}$.

$$\begin{aligned}\int x \ln x\, dx &= uv - \lim\ v\, du \\ &= \frac{x^2 \ln x}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \\ &= \frac{x^2 \ln x}{2} - \int \frac{x}{2} dx \\ &= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C\end{aligned}$$

2. Let $u = \ln x$ and $dv = \sqrt{x}$. Then $du = \frac{1}{x}dx$ and $v = \frac{2}{3}x^{\frac{3}{2}}$.

$$\begin{aligned}\int x \ln x\, dx &= uv - \int v\, du \\ \int_1^3 \sqrt{x} \ln x\, dx &= \frac{2x^{\frac{3}{2}} \ln x}{3} - \frac{2}{3}\int_1^3 x^{\frac{3}{2}} \cdot \frac{1}{x} dx \\ &= \frac{2x^{\frac{3}{2}} \ln x}{3} - \frac{2}{3}\int_1^3 x^{\frac{1}{2}} dx \\ &= \left(\frac{2x^{\frac{3}{2}} \ln x}{3} - \frac{4x^{\frac{3}{2}}}{9} \right) \Bigg|_1^3 \\ &= \frac{2 \times 3^{\frac{3}{2}} \ln x}{3} - \frac{4 \times 3^{\frac{3}{2}}}{9} - \left(0 - \frac{4}{9}\right) \\ &= \frac{2 \times 3^{\frac{3}{2}} \ln x}{3} - \frac{4 \times 3^{\frac{3}{2}}}{9} + \frac{4}{9}\end{aligned}$$

3. Let $u^2 = 2x + 1$. Then

$$\begin{aligned}u^2 &= 2x + 1 \\ u^2 - 1 &= 2x \\ \frac{u^2 - 1}{2} &= x \\ u\, du &= dx\end{aligned}$$

$$\begin{aligned}
\int \frac{x}{\sqrt{2x+1}}dx &= \int \frac{1}{u}\left(\frac{u^2-1}{2}\right) u\, du \\
&= \int \frac{u^2-1}{2}du \\
&= \frac{1}{2}\int \left(u^2-1\right) du \\
&= \frac{1}{2}\left[\frac{u^2}{3}-u\right] \\
&= \frac{1}{2}\left[\frac{\left(\sqrt{2x+1}\right)^3}{3}-\sqrt{2x+1}\right] \\
&= \frac{1}{2}\sqrt{2x+1}\left(\frac{2x+1}{3}-1\right)+C \\
&= \frac{1}{2}\cdot\frac{\sqrt{2x+1}\,(2x-2)}{3}+C \\
&= \frac{\sqrt{2x+1}\,(x-1)}{3}+C
\end{aligned}$$

4. Let $u = 1-x^2$. Then

$$\begin{aligned}
du &= -2x\, dx \\
-\frac{1}{2}du &= x\, dx
\end{aligned}$$

and

$$\begin{aligned}
u-1 &= -x^2 \\
-u+1 &= x^2
\end{aligned}$$

When using $u-$substitution, just put limits as u_1 and u_2 as placeholders on the integral. After u is replaced by the function of x, put back the original limits of integration.

$$\begin{aligned}
\int_0^1 x^3\sqrt{1-x^2}dx &= \int_{u_1}^{u_2} x^2\left(x\sqrt{1-x^2}\right)dx \\
&= \frac{1}{2}\int_{u_1}^{u_2} -(-u+1)\sqrt{u}\,du \\
&= \frac{1}{2}\int_{u_1}^{u_2}\left(u^{\frac{3}{2}}-u^{\frac{1}{2}}\right)du \\
&= \frac{1}{2}\left(\frac{u^{\frac{5}{2}}}{\frac{5}{2}}-\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) \\
&= \frac{1}{2}\left[\frac{2}{5}\left(1-x^2\right)^{\frac{3}{2}}-\frac{2}{3}\left(1-x^2\right)^{\frac{5}{2}}\right]\Bigg|_0^1 \\
&= 0-\frac{1}{2}\left(\frac{2}{5}-\frac{2}{3}\right) \\
&= \frac{2}{15}
\end{aligned}$$

5. Let $u = x$ and $dv = \cos\ x\ dx$. Then $du = dx$ and $v = \sin\ x$.

$$\begin{aligned}
\int x\ \cos x\ dx &= x\ \sin\ x-\int \sin\ x\ dx \\
&= x\ \sin\ x-(-\cos x)+C \\
&= x\ \sin\ x+\cos\ x+C
\end{aligned}$$

6. Let $u = x^3+9$. Then

$$\begin{aligned}
du &= 3x^2\ dx \\
\frac{1}{3}du &= x^2\ dx
\end{aligned}$$

$$\begin{aligned}
\int_0^1 x^2\sqrt{x^3+9}dx &= \int_{u_1}^{u_2}\frac{1}{3}\sqrt{u}\,du \\
&= \frac{1}{3}\int_{u_1}^{u_2} u^{\frac{1}{2}}du \\
&= \frac{1}{3}\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \\
&= \frac{2}{9}\sqrt{\left(x^3+9\right)^3}\Bigg|_0^1 \\
&= \frac{2}{9}\left(\sqrt[3]{10^2}-27\right)
\end{aligned}$$

7. Let

$$\begin{aligned} u &= \frac{1}{x} = x^{-1} \\ du &= -x^{-2}dx \\ -du &= \frac{1}{x^2}dx \end{aligned}$$

$$\begin{aligned} \int \left(\frac{1}{x^2} \times e^{\frac{1}{x}}\right) dx &= \int e^u \, du \\ &= -e^u + C \\ &= e^{\frac{1}{x}} + C \end{aligned}$$

8. Let $u = x^2$ and $dv = xe^{x^2}dx$. Then $du = 2x\,dx$ and $v = \frac{1}{2}e^{x^2}$.

$$\begin{aligned} \int x^3 e^{x^2} dx &= \frac{1}{2}x^2 e^{x^2} (2x)\, dx \\ &= \frac{1}{2}x^2 e^{x^2} - \int e^{x^2} (x)\, dx \\ &= \frac{1}{2}x^2 e^{x^2} - \frac{1}{2}e^{x^2} + C \end{aligned}$$

9. Let $u = \ln x$ and $dv = \frac{1}{x^{\frac{3}{2}}}dx = x^{-\frac{3}{2}}dx$. Then $du = \frac{1}{x}$ and $v = -\frac{2}{3}x^{\frac{-1}{2}}$.

$$\begin{aligned} \int \frac{\ln x}{x^{\frac{5}{2}}} dx &= \frac{-2}{3}\ln x \left(x^{\frac{-3}{2}}\right) - \int \left(\frac{-2}{3}\right) \frac{x^{\frac{-3}{2}}}{x} dx \\ &= \frac{-2\ln x}{3x^{\frac{-3}{2}}} + \frac{2}{3}\int x^{-\frac{5}{2}} dx \\ &= \frac{-2\ln x}{3x^{\frac{-3}{2}}} + \frac{2}{3}\frac{x^{-\frac{3}{2}}}{\left(-\frac{3}{2}\right)} + C \\ &= \frac{-2\ln x}{3x^{\frac{-3}{2}}} - \frac{4}{9x^{\frac{-3}{2}}} + C \end{aligned}$$

10. $\int\limits_1^e \frac{1}{x}dx = \ln x|_1^e = \ln e - \ln 0 = \ln e = 1$