7.4 Trigonometric Integrals

1. Let $u = \cos x$. Then $du = -\sin x\, dxx$.

$$\begin{aligned}\int \cos^4 x \sin x\, dx &= -\int u^4\, du\\ &= -\frac{u^5}{5} + C\\ &= -\frac{\cos^5 x}{5} + C\end{aligned}$$

2. Let $u = 5\varphi$. Then $du = 5d\varphi$.

$$\begin{aligned}\int \sin^2 5\varphi d\varphi &= \frac{1}{5}\int \sin^2 u\, du\\ &= \frac{1}{5}\int \frac{1}{2}(1-\cos 2u)du\\ &= \frac{1}{10}\left(u - \frac{1}{2}\sin 2u\right) + C\\ &= \frac{1}{10}\left(5\varphi - \frac{1}{2}\sin(2\times 5\varphi)\right) + C\\ &= \frac{1}{2}\varphi - \frac{1}{20}\sin(10\varphi) + C\end{aligned}$$

3. Let $u = \sin 2z$. Then $du = 2\cos 2z\, dz$.

$$\begin{aligned}\int \sin^2 2z \cos^3 2z\, dz &= \int \sin^2 2z \cos^3 2z(1-\sin^2 2z)dz\\ &= \int \sin^2 2z \cos 2z\, dz - \int \sin^4 2z \cos 2z\, dz\\ &= \frac{1}{2}\int u^2 du - \frac{1}{2}\int u^4 du\\ &= \frac{u^3}{6} - \frac{u^5}{10} + C\\ &= \frac{\sin^3 2z}{6} - \frac{\sin^5 2z}{10} + C\end{aligned}$$

4.

$$\begin{aligned}\int \sin x \cos\left(\frac{x}{2}\right) dx &= \int 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) dx\\ &= \int 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) dx\end{aligned}$$

Let $u = \cos\left(\frac{x}{2}\right)$. Then $du = -\frac{1}{2}\sin\left(\frac{x}{2}\right) dx$.

$$\begin{aligned}\int 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx &= -4\int u^2 du\\ &= -\frac{4u^3}{3}+C\\ &= -\frac{4}{3}\cos^3\left(\frac{x}{2}\right)+C\end{aligned}$$

5. Let $u=\sec x$. Then $du=\sec x\tan x\ dx$.

$$\begin{aligned}\int \sec^4 x\tan^3 x dx &= \int \tan^2 x\sec^3 x(\sec x\tan x)dx\\ &= \int (\sec^2 x-1)\sec^3 x(\sec x\tan x)dx\\ &= \int \sec^5 x(\sec x\tan x)dx-\int \sec^3 x(\sec x\tan x)dx\\ &= \int u^5 du-\int u^3 du\\ &= \frac{u^6}{6}-\frac{u^4}{4}+C\\ &= \frac{\sec^6 x}{6}-\frac{\sec^4 x}{4}+C\end{aligned}$$

6.

$$\begin{aligned}\int \tan^4 x\sec x dx &= \int (\sec^2 x-1)(\sec^2 x-1)\sec x\ dx\\ &= \int (\sec^5 x-2\sec^3 x+\sec x)dx\\ &= \frac{\sec^3 x\tan x}{4}+\frac{3}{4}\int \sec^3 x\ dx-2\frac{\sec x\tan x}{2}-2\times\frac{1}{2}\int \sec x\ dx+\ln|\sec x+\tan x|+C\\ &= \frac{\sec^3 x\tan x}{4}+\frac{3}{4}\left(\frac{\sec x\tan x}{2}+\frac{1}{2}\int \sec x\ dx\right)-2\frac{\sec x\tan x}{2}-\ln|\sec x+\tan x|+-\ln|\sec x+\tan x|+C\\ &= \frac{\sec^3 x\tan x}{4}-\frac{5\sec x\tan x}{8}+\frac{3}{8}\ln|\sec x+\tan x|\end{aligned}$$

7.

$$\begin{aligned}\int \sqrt{\tan x}\sec^4 x\ dx &= \int \tan^{\frac{1}{2}} x\sec^4 x\ dx\\ &= \int \tan^{\frac{1}{2}} x(\tan^2 x+1)\sec^2 x dx\\ &= \int \left[\tan^{\frac{5}{2}} x\sec^2 x+\tan^{\frac{1}{2}} x\sec^2 x\right]dx\end{aligned}$$

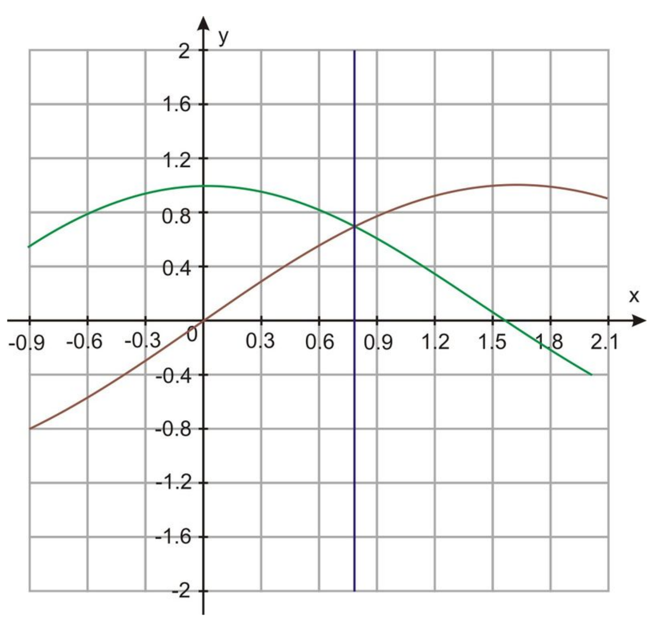
Let $u=\tan x$. Then $du=\sec^2 x\ dx$.

$$\int \left[\tan^{\frac{5}{2}} x \sec^2 x + \tan^{\frac{1}{2}} x \sec^2 x\right] dx = \int u^{\frac{5}{2}} du + \int u^{\frac{1}{2}} du$$
$$= \frac{u^{\frac{7}{2}}}{\frac{7}{2}} + \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + C$$
$$= \frac{2\tan^{\frac{7}{2}} x}{7} + \frac{2\tan^{\frac{3}{2}} x}{3} + C$$

8. Let $u = \frac{x}{2}$. Then $du = \frac{1}{2}dx$. If $x = 0$, then $u = 0$. If $x = \frac{\pi}{2}$, then $u = \frac{\pi}{4}$.

$$\int_0^{\frac{\pi}{2}} \tan^5\left(\frac{x}{2}\right) dx = \int_0^{\frac{\pi}{4}} 2\tan^5 u \, du$$
$$= \frac{2\tan^4 u}{4}\Big|_0^{\frac{\pi}{4}} - 2\int_0^{\frac{\pi}{4}} \tan^3 u \, du$$
$$= 2\left(\frac{\sqrt{2}}{2}\right)^4 - \frac{2\tan^4 u}{2}\Big|_0^{\frac{\pi}{4}} + 2\int_0^{\frac{\pi}{2}} \tan u \, du$$
$$= \frac{8}{16} - \left(\frac{\sqrt{2}}{2}\right)^2 - 2\ln|\cos u|_0^{\frac{\pi}{4}}$$
$$= \frac{8}{16} - 1 - 2\ln\left|\cos\left(\frac{\pi}{4}\right)\right| - 0$$
$$= -\frac{1}{2} - 2\ln\left|\frac{\sqrt{2}}{2}\right|$$
$$= -\frac{1}{2} + \ln\left|\frac{4}{2}\right|$$
$$= -\frac{1}{2} + \ln 2$$

9.



$$\begin{aligned}
V &= \pi\left(\int_0^{\frac{\pi}{4}} \cos^2 x\,dx - \int_0^{\frac{\pi}{4}} \sin^2 x\,dx\right)\\
&= \pi\left[\frac{1}{2}\cos x \sin x + \frac{x}{2}\right]_0^{\frac{\pi}{4}} - \pi\left[-\frac{1}{2}\cos x \sin x + \frac{x}{2}\right]_0^{\frac{\pi}{4}}\\
&= \pi\left[\left(\frac{1}{4}+\frac{\pi}{8}\right) - \left(-\frac{1}{4}+\frac{\pi}{8}\right)\right]\\
&= \pi\left(\frac{1}{2}\right)\\
&= \frac{\pi}{2}
\end{aligned}$$

10.

a. $\int \csc x\,dx = \int \csc x \frac{\csc x + \cot x}{\csc x + \cot x} dx$

Let $u = \csc x + \cot x$. Then $du = (-\csc x \cot x - \csc^2 x)dx$.

$$\begin{aligned}
\int \csc x dx &= -\int \frac{1}{u} du\\
&= -\ln|u| + C\\
&= -\ln|\csc x + \cot x| + C
\end{aligned}$$

b.

$$\begin{aligned}
\int \csc x\,dx &= \int \frac{1}{\sin x} dx\\
&= \int \frac{\sin x}{\sin^2 x} dx\\
&= \int \frac{\sin x}{(1-\cos x)(1+\cos x)} dx
\end{aligned}$$

Let $u = \cos x$. Then $du = -\sin x\,dx$.

$$\begin{aligned}
\int \frac{\sin x}{(1-\cos x)(1+\cos x)}dx &= -\int \frac{du}{(1-u)(1+u)} \\
&= \int \frac{A}{1-u}du + \int \frac{B}{1+u}du \\
&= -\int \frac{\frac{1}{2}}{1-u}du - \int \frac{\frac{1}{2}}{1+u}du \\
&= \int \frac{\frac{1}{2}}{1-\cos x}du - \int \frac{\frac{1}{2}}{1+\cos x}du \\
&= \frac{1}{2}|1-\cos x| - \frac{1}{2}|1+\cos x| + C \\
&= \frac{1}{2}\left|\frac{1-\cos x}{1+\cos x}\right| + C \\
&= \ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C \\
&= \ln\left|\tan\left(\frac{x}{2}\right)\right| + C
\end{aligned}$$

OR

$$\begin{aligned}
\ln\sqrt{\frac{1-\cos x}{1+\cos x}} + C &= \ln\sqrt{\frac{(1-\cos x)}{(1+\cos x)}\cdot\frac{(1-\cos x)}{(1-\cos x)}} + C \\
&= \ln\sqrt{\frac{(1-\cos x)^2}{1-\cos^2 x}} + C \\
&= \ln\left|\frac{1-\cos x}{\sin x}\right| + C \\
&= \ln\left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| + C \\
&= \ln|\csc x - \cot x| + C
\end{aligned}$$