

4.4 Definite Integrals

1. Each sub-interval of $[0, 2]$ will have length of $\Delta x = \frac{2-0}{4} = \frac{1}{2}$. The midpoints are $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}$.

$$\begin{aligned} R_4 &= \left(2 - \frac{1}{4}\right) \frac{1}{2} + \left(2 - \frac{3}{4}\right) \frac{1}{2} + \left(2 - \frac{5}{4}\right) \frac{1}{2} + \left(2 - \frac{7}{4}\right) \frac{1}{2} \\ &= \left(\frac{7}{4}\right) \frac{1}{2} + \left(\frac{5}{4}\right) \frac{1}{2} + \left(\frac{3}{4}\right) \frac{1}{2} + \left(\frac{1}{4}\right) \frac{1}{2} \\ &= \frac{7+5+3+1}{8} \\ &= 2 \end{aligned}$$

2. From 0 to 2, the line forms a triangle with the x -axis from $(0, 2)$ to $(2, 0)$. The height of the triangle is 2. The width is $2 - 0 = 2$. The area of the triangle is $\frac{1}{2}(2)(2) = 2$.

3. Use the right endpoints to compute the integral. Divide $[0, 2]$ into n subintervals of length $\Delta x = \frac{2-0}{n} = \frac{2}{n}$. Use the right endpoints: $\frac{2}{n}, \frac{4}{n}, K, \frac{2i}{n}, K, \frac{2n}{n}$.

$$\begin{aligned} \int_0^2 (2-x) dx &= \lim_{n \rightarrow \infty} \sum_{i=1}^n f\left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(2 - \frac{2i}{n}\right) \left(\frac{2}{n}\right) \\ &= \lim_{n \rightarrow \infty} \left[\sum_{i=1}^n \frac{4}{n} - \sum_{i=1}^n \left(\frac{2i}{n}\right) \left(\frac{2}{n}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\sum_{i=1}^n \frac{4}{n} - \sum_{i=1}^n \left(\frac{4i}{n}\right) \right] \\ &= \lim_{n \rightarrow \infty} \left[\sum_{i=1}^n \frac{4}{n} - \frac{4}{n^2} \sum_{i=1}^n i \right] \\ &= \lim_{n \rightarrow \infty} \left[\frac{4n}{n} - \frac{4}{n^2} \frac{n^2+n}{2} \right] \\ &= \lim_{n \rightarrow \infty} \left[4 - 2 - \frac{2}{n} \right] = \lim_{n \rightarrow \infty} \left[2 - \frac{2}{n} \right] = 2 \end{aligned}$$

4. Each sub-interval of $[1, 4]$ will have length of $\Delta x = \frac{4-1}{5} = \frac{3}{5}$. The left endpoints are $1, 1 + \frac{3}{5} = \frac{8}{5}, 1 + \frac{6}{5} = \frac{11}{5}, 1 + \frac{9}{5} = \frac{14}{5}, 1 + \frac{12}{5} = \frac{17}{5}$.

$$\begin{aligned}
 R_5 &= (1^2 - 1) \frac{3}{5} + \left(\left(\frac{8}{5} \right)^2 - \frac{8}{5} \right) \frac{3}{5} + \left(\left(\frac{11}{5} \right)^2 - \frac{11}{5} \right) \frac{3}{5} + \left(\left(\frac{14}{5} \right)^2 - \frac{14}{5} \right) \frac{3}{5} + \left(\left(\frac{17}{5} \right)^2 - \frac{17}{5} \right) \frac{3}{5} \\
 &= 0 + (2.56 - 1.6) \frac{3}{5} + (4.84 - 2.2) \frac{3}{5} + (7.84 - 2.8) \frac{3}{5} + (11.56 - 3.4) \frac{3}{5} \\
 &= 0.576 + 1.584 + 3.024 + 4.896 \\
 &= 10.08
 \end{aligned}$$

$$5. \int_1^4 (x^2 - x) dx \lim_{n \rightarrow \infty} \sum_{i=1}^n f \left(1 + \frac{3i}{n} \right) \left(\frac{3}{n} \right)$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left[\left(1 + \frac{3i}{n} \right)^2 - \left(1 + \frac{3i}{n} \right) \right] \left(\frac{3}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(1 + \frac{6i}{n} + \frac{9i^2}{n^2} - 1 - \frac{3i}{n} \right) \left(\frac{3}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{3i}{n} + \frac{9i^2}{n^2} \right) \left(\frac{3}{n} \right) \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \right) \left[\sum_{i=1}^n \frac{3i}{n} + \sum_{i=1}^n \frac{9i^2}{n^2} \right] \\
 &= \lim_{n \rightarrow \infty} \left(\frac{3}{n} \right) \left[\frac{3}{n} \sum_{i=1}^n i + \frac{9}{n^2} \sum_{i=1}^n i^2 \right] \\
 &= \lim_{n \rightarrow \infty} \left[\left(\frac{9}{n^2} \right) \left(\frac{n^2 + n}{2} \right) + \left(\frac{27}{n^3} \right) \left(\frac{2n^3 + 3n^2 + n}{6} \right) \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{9n^2 + 9n}{2n^2} + \left(\frac{18n^3 + 27n^2 + 9n}{2n^3} \right) \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{9}{2} + \frac{9}{2n} + 9 - \frac{27}{2n^3} + \frac{9}{2n^2} \right] \\
 &= 13.5
 \end{aligned}$$

6. a. Each sub-interval of $[0, 1]$ will have length of $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The right endpoints are $\frac{1}{2}, 1$.

$$\begin{aligned}
 R_2 &= \left(3 \left(\frac{1}{2} \right)^2 \right) \frac{1}{2} + \left(3(1)^2 \right) \frac{1}{2} \\
 &= \left(\frac{3}{4} \right) \frac{1}{2} + (3) \frac{1}{2} \\
 &= \frac{3}{8} + \frac{3}{2} \\
 &= 1.875
 \end{aligned}$$

b. Each sub-interval of $[0, 1]$ will have length of $\Delta x = \frac{1-0}{5} = \frac{1}{5}$. The right endpoints are $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$.

$$\begin{aligned}
 R_5 &= \left(3\left(\frac{1}{5}\right)^2\right)\frac{1}{5} + \left(3\left(\frac{2}{5}\right)^2\right)\frac{1}{5} + \left(3\left(\frac{3}{5}\right)^2\right)\frac{1}{5} + \left(3\left(\frac{4}{5}\right)^2\right)\frac{1}{5} + \left(3(1)^2\right)\frac{1}{5} \\
 &= \frac{3}{5}\left(\frac{1+4+9+16}{25} + 1\right) \\
 &= \frac{3}{5}(2.2) \\
 &= 1.32
 \end{aligned}$$

c. Each sub-interval of $[0, 1]$ will have length of $\Delta x = \frac{1-0}{10} = \frac{1}{10}$. The right endpoints are $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1$.

$$\begin{aligned}
 R_{10} &= \left(3\left(\frac{1}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{2}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{3}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{4}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{5}{10}\right)^2\right)\frac{1}{10} \\
 &\quad + \left(3\left(\frac{6}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{7}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{8}{10}\right)^2\right)\frac{1}{10} + \left(3\left(\frac{9}{10}\right)^2\right)\frac{1}{10} + \left(3(1)^2\right)\frac{1}{10} \\
 &= \frac{3}{10}\left(\frac{1+4+9+16+25+36+49+64+81}{100} + 1\right) \\
 &= \frac{3}{10}\left(\frac{285}{100} + 1\right) \\
 &= 1.15
 \end{aligned}$$

d. The area should be 1.

7. a. Each sub-interval of $[0, 1]$ will have length of $\Delta x = \frac{1-0}{2} = \frac{1}{2}$. The right endpoints are $\frac{1}{2}, 1$.

$$\begin{aligned}
 R_2 &= (e^{0.5})\frac{1}{2} + (e^1)\frac{1}{2} \\
 &= (1.649)\frac{1}{2} + (2.718)\frac{1}{2} \\
 &= 2.18
 \end{aligned}$$

b. Each sub-interval of $[0, 1]$ will have length of $\Delta x = \frac{1-0}{5} = \frac{1}{5}$. The right endpoints are $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$.

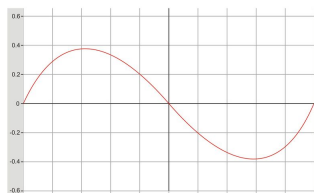
$$\begin{aligned}
 R_5 &= \left(e^{\frac{1}{5}}\right)\frac{1}{5} + \left(e^{\frac{2}{5}}\right)\frac{1}{5} + \left(e^{\frac{3}{5}}\right)\frac{1}{5} + \left(e^{\frac{4}{5}}\right)\frac{1}{5} + (e^1)\frac{1}{5} \\
 &= \frac{1}{5}(1.221 + 1.492 + 1.822 + 2.226 + 2.718) \\
 &= \frac{1}{5}(9.479) \\
 &= 1.896
 \end{aligned}$$

c. Each sub-interval of $[0, 1]$ will have length of $\Delta x = \frac{1-0}{10} = \frac{1}{10}$. The right endpoints are $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}, \frac{5}{10}, \frac{6}{10}, \frac{7}{10}, \frac{8}{10}, \frac{9}{10}, 1$.

$$\begin{aligned}
 R_{10} &= \left(e^{\frac{1}{10}}\right) \frac{1}{10} + \left(e^{\frac{2}{10}}\right) \frac{1}{10} + \left(e^{\frac{3}{10}}\right) \frac{1}{10} + \left(e^{\frac{4}{10}}\right) \frac{1}{10} + \left(e^{\frac{5}{10}}\right) \frac{1}{10} \\
 &+ \left(e^{\frac{6}{10}}\right) \frac{1}{10} + \left(e^{\frac{7}{10}}\right) \frac{1}{10} + \left(e^{\frac{8}{10}}\right) \frac{1}{10} + \left(e^{\frac{9}{10}}\right) \frac{1}{10} + (e^1) \frac{1}{10} \\
 &= \frac{1}{10} (1.105 + 1.221 + 1.35 + 1.492 + 1.649 + 1.822 + 2.014 + 2.226 + 2.46 + 2.718) \\
 &= \frac{1}{10} (18.06) \\
 &= 1.81
 \end{aligned}$$

d. The area should be $e - 1 \approx 1.7$.

8.



The graph is symmetric around the origin. The area is 0.

9.

$$\begin{aligned}
 2 \int_{-1}^0 (x^3 - x) dx &= 2 \left[\int_{-1}^0 x^3 dx - \int_{-1}^0 x dx \right] \\
 &= 2 \left[\lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{(-i)^3}{n^3} \right) \frac{1}{n} - \lim_{n \rightarrow \infty} \sum_{i=1}^n \left(\frac{-i}{n} \right) \frac{1}{n} \right] \\
 &= 2 \left[\lim_{n \rightarrow \infty} \left(-\frac{1}{n^4} \sum_{i=1}^n i^3 + \frac{1}{n^2} \sum_{i=1}^n i \right) \right] \\
 &= 2 \left[\lim_{n \rightarrow \infty} \left(-\frac{1}{n^4} \left(\frac{n(n+1)}{2} \right)^2 + \frac{1}{n^2} \left(\frac{n(n+1)}{2} \right) \right) \right] \\
 &= 2 \left[\lim_{n \rightarrow \infty} \left(\left(-\frac{1}{n^2} \times \frac{n^2 + 2n + 1}{4} \right) + \frac{n+1}{2n} \right) \right] \\
 &= 2 \left[\lim_{n \rightarrow \infty} \left(-\frac{1}{4} - \frac{1}{2n} - \frac{1}{4n^2} + \frac{1}{2} + \frac{1}{2n} \right) \right] \\
 &= 2 \left(-\frac{1}{4} + \frac{1}{2} \right) \\
 &= 2 \left(\frac{1}{4} \right) = \frac{1}{2}
 \end{aligned}$$

10.

$$\begin{aligned}y &= \sqrt{9-x^2} \\y^2 &= 9-x^2 \\x^2+y^2 &= 9 \\x^2+y^2 &= 3^2\end{aligned}$$

From $x = 0$ to $x = 3$, $y = \sqrt{9-x^2}$ is a quarter of the circle around the origin with radius 3. The area is $\frac{\pi r^2}{4} = \frac{9\pi}{4}$.