6.006	Lecture 20	Nov. 29, 2011

TODAY: Dynamic Programming II (of 4)
- 5 easy steps
- text justification
- perfect-information Blackjack
- parent pointers

Summary:
* DP ≈ "careful brute force"
* DP ≈ guessing + recursion + memoization
* DP ≈ dividing into reasonable # subproblems whose solutions relate — acyclicly — usually via guessing parts of solution

* time = # subproblems · $\underbrace{\text{time/subproblem}}$
  treating recursive calls as $O(1)$
  (usually mainly guessing)
  - essentially an amortization
  - count each subproblem only once; after first time, costs $O(1)$ via memoization

* DP ≈ shortest paths in some DAG

* 5 easy steps to dynamic programming:

1. define subproblems — count # subprobs.
2. guess (part of solution) — count # choices
3. relate subprob. solutions — compute time/subprob.
4. recurse + memoize — time = time/subprob. · # subprobs.

OR build DP table bottom-up
- check subprobs. acyclic/topological order

5. solve original problem: = a subproblem

OR by combining subprob. solutions ($\Rightarrow$ extra time)

Examples:	Fibonacci	Shortest Paths
1. subprobs:	F_k for $1 \le k \le n$	$\delta_k(s,v)$ for $v \in V$, $0 \le k < \lvert V \rvert$ = min. $s \to v$ path using $\le k$ edges
# subprobs:	n	V^2
2. guess:	nothing	edge into v (if any)
# choices:	1	indegree$(v)+1$
3. recurrence:	$F_k = F_{k-1} + F_{k-2}$	$\delta_k(s,v) = \min\{\delta_{k-1}(s,u) + w(u,v) \mid (u,v) \in E\}$
time/subprob.:	$\Theta(1)$	$\Theta(1+\text{indegree}(v))$
4. topo. order:	for $k=1,\dots,n$	for $k=\emptyset,1,\dots,\lvert V \rvert-1$ for $v \in V$
total time:	$\Theta(n)$	$\Theta(VE)$ $+\Theta(V^2)$ unless efficient about indeg. $\emptyset$
5. orig. prob.:	F_n	$\delta_{\lvert V \rvert-1}(s,v)$ for $v \in V$
extra time:	$\Theta(1)$	$\Theta(V)$

Text justification: split text into "good" lines

- obvious (MS Word/OpenOffice) algorithm: put as many words fit on first line, repeat
- but this can make very bad lines:

```
blah blah blah              blah     blah
b    l    a    h     vs.    blah     blah
reallylongword              reallylongword
```

- define badness(i, j) for line of words[i:j]
  e.g. $\begin{cases} \infty & \text{if total length} > \text{page width} \\ (\text{page width} - \text{total length})^3 & \text{else} \end{cases}$
- goal: split words into lines to min. $\sum_i$ badness

① subproblem = min. badness for suffix words[i:]
$\Rightarrow$ # subproblems = $\Theta(n)$ where n = # words

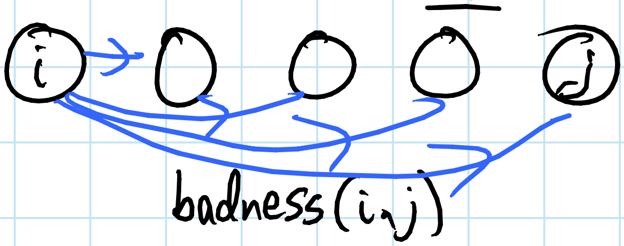
② guessing = where to end first line, say i:j
$\Rightarrow$ # choices = $n - i = O(n)$

③ recurrence:
- $DP[i] = \min(\text{badness}(i, j) + DP[j]$
  for j in range(i+1, n+1))
- $DP[n] = \emptyset$

$\Rightarrow$ time per subproblem = $\Theta(n)$

④ order: for $i = n, n-1, \ldots, 1, \emptyset$
total time = $\Theta(n^2)$

DAG:



⑤ solution = $DP[\emptyset]$

Perfect-information Blackjack:

- given entire deck order: $c_0, c_1, \ldots, c_{n-1}$
- 1-player game against stand-on-17 dealer
- when should you hit or stand? GUESS
- goal: maximize winnings for fixed bet \$1
- may benefit from losing one hand to improve future hands!

(1) subproblems: $BJ(i) = \text{best play of } \overbrace{c_i, \ldots, c_{n-1}}^{\text{remaining cards}}$

$\uparrow$ # cards "already played"

$\Rightarrow$ # subproblems $= n$

(2) guess: how many times player "hits" $\leftarrow$ draws another card

$\Rightarrow$ # choices $\leq n$

(3) recurrence: $BJ(i) = \max($

$O(n) \rightarrow$ outcome $\in \{+1, 0, -1\}$ + $BJ(i + \#\text{cards used})$

$O(n) \rightarrow$ for #hits in $0, 1, \ldots$

if valid play ~ don't hit after bust $)$

$\Rightarrow$ time/subproblem $= \Theta(n^2)$

(4) order: for i in reversed(range(n))

- total time $= \Theta(n^3)$

[time is really $\sum_{i=0}^{n-1} \sum_{\#h=0}^{n-i-O(1)} \Theta(n-i-\#h) = \Theta(n^3)$ still]

(5) solution $= BJ(0)$

– detailed recurrence: (before memoization)
(ignoring splits/betting)

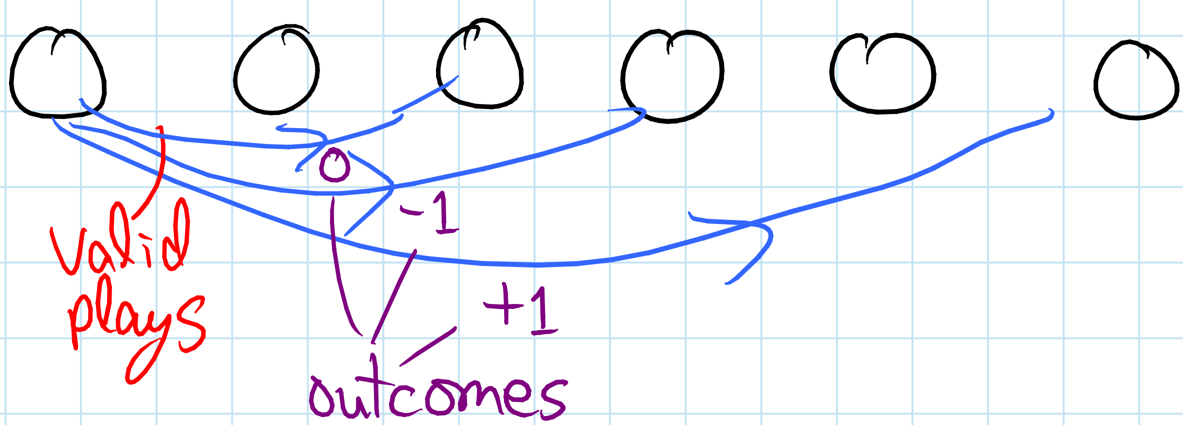
```
BJ(i):
    if n-i < 4: return 0                          (not enough cards)
    for p in range(2, n-i-1):                     (# cards taken)
        player = sum(c_i, c_i+2, c_i+4:i+p+2)
        if player > 21:                           (bust)
            options.append(-1 + BJ(i+p+2))
            break                                 ↳ bust
        for d in range(2, n-i-p):
            dealer = sum(c_i+1, c_i+3, c_i+p+2:i+p+d)
            if dealer ≥ 17: break
        if dealer > 21: dealer = 0                (bust)
        options.append(cmp(player, dealer) +
                                        BJ(i+p+d))
    return max(options)
```

$\Theta(n^2)$ — whole `for p` loop

$\Theta(n)$ — `player = sum(...)`

$\Theta(n)$ with care — `for d` loop

DAG view:



Parent pointers:

to recover actual solution in addition to cost, store parent pointers (which guess used at each subproblem) & walk back

- typically: remember argmin/argmax in addition to min/max
- e.g. text justification:

(3′) $DP[i] = \min\Big(\big(\text{badness}(i,j) + DP[j][\emptyset],\ j\big)$ for j in range$(i+1, n+1)\Big)$

$DP[n] = (\emptyset, \text{None})$

(5′)
```
i = 0
while i is not None:
    start line before word i
    i = DP[i][1]
```

- just like memoization & bottom-up, this transformation is automatic (no thinking required)

6.006 Introduction to Algorithms
Fall 2011